The Gorensteinness of the endomorphism rings of ideals

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based on the recent works jointly with

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1. Introduction

This talk is based on the recent research below.

- *When are the rings $I : I$ Gorenstein?*, arXiv:2111.13338
  (with S. Goto, S.-i. Iai, and N. Matsuoka)

Let $(R, \mathfrak{m})$ be a Noetherian local ring with depth $R > 0$.

Problem 1.1

When is $\mathrm{End}_R(\mathfrak{m}) = \mathfrak{m} : \mathfrak{m}$ Gorenstein?

- If $\dim R = 1$, then

$\mathfrak{m} : \mathfrak{m}$ is a Gorenstein ring $\iff$ R is almost Gorenstein ring and $v(R) = \mathrm{e}(R)$

- If $\dim R \geq 2$, then $\operatorname{depth} R \geq 2$ if and only if $\mathfrak{m} : \mathfrak{m} = R$.
  Hence, provided $\operatorname{depth} R \geq 2$,

$\mathfrak{m} : \mathfrak{m}$ is a Gorenstein ring $\iff$ R is a Gorenstein ring.

How about the case where $\dim R \geq 2$ and $\operatorname{depth} R = 1$?

Key ideas

- (S_2)-ifications
- trace ideals

- If $\dim R = 1$, then

$$\mathfrak{m} : \mathfrak{m} \text{ is a Gorenstein ring } \iff R \text{ is almost Gorenstein ring and } v(R) = \mathrm{e}(R)$$

- If $\dim R \geq 2$, then depth $R \geq 2$ if and only if $\mathfrak{m} : \mathfrak{m} = R$.
  Hence, provided depth $R \geq 2$,

$$\mathfrak{m} : \mathfrak{m} \text{ is a Gorenstein ring } \iff R \text{ is a Gorenstein ring.}$$

How about the case where $\dim R \geq 2$ and depth $R = 1$?

Key ideas

- (S_2)-ifications
- trace ideals

2. Basic results on (S_2)-ifications

Throughout this talk, let

- R an arbitrary commutative Noetherian ring
- $\mathrm{Q}(R)$ the total ring of fractions of R
- $\mathrm{Ht}_{\geq 2}(R) = \{I \mid I \text{ is an ideal of } R,\ \mathrm{ht}_R I \geq 2\}$
- $W(R) = \{a \in R \mid a \text{ is a non-zerodivisor on } R\}$

We fix a $\mathrm{Q}(R)$-module V and an R-submodule M of V.

Define

$$M \subseteq \widetilde{M} = \{f \in V \mid If \subseteq M \text{ for some } I \in \mathrm{Ht}_{\geq 2}(R)\} \subseteq V.$$

- If L is an R-submodule of V and $M \subseteq L$, then $\widetilde{M} \subseteq \widetilde{L}$.
- $\widetilde{R}$ considered inside $\mathrm{Q}(R)$ is an intermediate ring $R \subseteq \widetilde{R} \subseteq \mathrm{Q}(R)$.
- $\widetilde{M}$ is an $\widetilde{R}$-submodule of V.

Let $a, b \in R$ and N an R-module. The pair a, b is called N-sequence, if

$$a \text{ is a } N\text{-NZD and } b \text{ is a } N/aN\text{-NZD.}$$

Here, we don't require $N/(a,b)N \neq (0)$.

Lemma 2.1

Let $a, b \in W(R)$. If $\mathrm{ht}_R(a,b) \geq 2$, then the pair a, b is $\widetilde{M}$-sequence.

(Proof) Let $f \in \widetilde{M}$ and assume $bf = ag$ for some $g \in \widetilde{M}$. Set $x = \frac{f}{a} = \frac{g}{b}$, and choose $I, J \in \mathrm{Ht}_{\geq 2}(R)$ so that $If + Jg \subseteq M$. Then

$$(Ia + Jb)x \subseteq M$$

whence $x \in \widetilde{M}$ because $Ia + Jb \in \mathrm{Ht}_{\geq 2}(R)$. □

Proposition 2.2

Suppose that one of the following conditions is satisfied.

(1) $\mathrm{Q}(R)M = V$.

(2) $\mathrm{ht}_R\,\mathfrak{p} \le 1$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}\,R$.

Then

$$M = \widetilde{M} \iff \text{every pair } a, b \in W(R) \text{ with } \mathrm{ht}_R(a,b) \ge 2 \text{ is } M\text{-sequence.}$$

(Proof) Assume $M \ne \widetilde{M}$ and consider $Z = \widetilde{M}/M$. Let $\mathfrak{p} \in \mathrm{Ass}_R\, Z$ and write $\mathfrak{p} = M :_R f$ for some $f \in \widetilde{M} \setminus M$. Choose $I \in \mathrm{Ht}_{\ge 2}(R)$ s.t. $If \subseteq M$. Then $I \subseteq \mathfrak{p}$. Notice that

$$af \in M \text{ for some } a \in W(R).$$

Therefore, $\mathrm{ht}_R(a,b) \ge 2$ for some $b \in W(R) \cap \mathfrak{p}$, whence a, b is M-sequence. So

$$0 \to (0) :_Z a \xrightarrow{\sigma} M/aM \to \widetilde{M}/a\widetilde{M} \to Z/aZ \to 0$$

where $b\sigma(\overline{f}) = \sigma(\overline{bf}) = 0$, because $bf \in M$. Thus $\sigma(\overline{f}) = 0$, so that $f \in M$. This is impossible.

Corollary 2.3

Suppose that one of the following conditions is satisfied.

(1) $\mathrm{Q}(R)M = V$.

(2) $\mathrm{ht}_R\, \mathfrak{p} \le 1$ *for* $\forall \mathfrak{p} \in \mathrm{Ass}\, R$.

Then the following assertions hold true.

(a) $\widetilde{\widetilde{M}} = \widetilde{M}$.

(b) *Let* $M \subseteq L \subseteq V$ *be an* R*-submodule of* V. *If every pair* $a, b \in W(R)$ *with* $\mathrm{ht}_R(a, b) \ge 2$ *is* L*-sequence, then* $\widetilde{M} \subseteq \widetilde{L} = L$.

Recall that a finitely generated R-module N satisfies (S_n), if

$$\operatorname{depth}_{R_\mathfrak{p}} N_\mathfrak{p} \geq \min\{n, \dim R_\mathfrak{p}\} \text{ for } \forall \mathfrak{p} \in \operatorname{Supp}_R N.$$

Theorem 2.4

Suppose R satisfies (S_1). If $\widetilde{M}$ is a finitely generated R-module, then $\widetilde{M}$ is the smallest R-submodule of V which contains M and satisfies (S_2).

Corollary 2.5

Suppose R satisfies (S_1). If $\widetilde{R}$ is a finitely generated R-module, then $\widetilde{R}$ is the smallest module-finite birational extension of R satisfying (S_2).

Corollary 2.6

If R satisfies (S_2), then $R = \widetilde{R}$.

In the rest of this section, we assume

- M is a finitely generated R-module,
- $\mathrm{Q}(R)M = V$, and
- $(0) :_{\mathrm{Q}(R)} V = (0)$.

Note that, every $f \in V$ has the form $f = \frac{m}{a}$ with $a \in W(R)$ and $m \in M$.

Let $a \in W(R)$ and let

$$aM = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_R M/aM} Q(\mathfrak{p})$$

be a primary decomposition of aM in M. Set

$$\mathrm{U}(aM) = \begin{cases} M & \text{if } aM = M, \\ \bigcap_{\mathfrak{p} \in \mathrm{Min}_R M/aM} Q(\mathfrak{p}) & \text{if } aM \neq M. \end{cases}$$

Theorem 2.7

Let $a \in W(R)$ and $m \in M$. Then $\frac{m}{a} \in \widetilde{M}$ if and only if $m \in \mathrm{U}(aM)$.

(Proof) May assume $aM \neq M$. Suppose $\frac{m}{a} \in \widetilde{M}$ and choose $I \in \mathrm{Ht}_{\geq 2}(R)$ so that $I \subseteq aM :_R m$. Let $\mathfrak{p} \in \mathrm{Min}_R\, M/aM$. Since $\mathrm{ht}_R\, \mathfrak{p} = 1$, $aM :_R m \not\subseteq \mathfrak{p}$, whence

$$m \in [aM]_{\mathfrak{p}} \cap M = Q(\mathfrak{p}).$$

Hence, $m \in \mathrm{U}(aM)$.

Conversely, suppose $m \in \mathrm{U}(aM)$. If $aM = \mathrm{U}(aM)$, then $m \in aM$, so $\frac{m}{a} \in \widetilde{M}$. May assume $aM \neq \mathrm{U}(aM)$. Consider $\mathcal{F} = (\mathrm{Ass}_R\, M/aM) \setminus (\mathrm{Min}_R\, M/aM)$. Then, $\mathcal{F} \neq \emptyset$ and for each $\mathfrak{p} \in \mathcal{F}$,

$$\exists\ \ell = \ell(\mathfrak{p}) \gg 0 \ \text{ s.t. } \ \mathfrak{p}^{\ell} M \subseteq Q(\mathfrak{p}).$$

By setting $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}^{\ell(\mathfrak{p})} \in \mathrm{Ht}_{\geq 2}(R)$, we have

$$\mathfrak{a}\, \mathrm{U}(aM) \subseteq \bigcap_{\mathfrak{p} \in \mathcal{F}} Q(\mathfrak{p}) \cap \mathrm{U}(aM) = aM.$$

Hence $\frac{\mathrm{U}(aM)}{a} \subseteq \widetilde{M}$, as desired. □

Therefore, if $\widetilde{M} \subseteq \frac{M}{a}$ for some $a \in W(R)$, then $\widetilde{M} = \frac{\mathrm{U}(aM)}{a}$.

3. Trace ideals

Let M, X be R-modules. Consider the homomorphism

$$\tau : \mathrm{Hom}_R(M, X) \otimes_R M \to X, \;\; f \otimes m \mapsto f(m)$$

where $f \in \mathrm{Hom}_R(M, X)$ and $m \in M$.

We set $\mathrm{Tr}_X(M) = \mathrm{Im}\,\tau$ and call it the trace module of M in X.

Proposition 3.1 (Lindo)

Let I be an ideal of R. Then TFAE.

(1) *I is a trace ideal in R, i.e., $I = \mathrm{Tr}_R(M)$ for some R-module M.*

(2) $I = \mathrm{Tr}_R(I)$.

(3) *The embedding $\iota : I \to R$ induces $\mathrm{Hom}_R(I, I) \cong \mathrm{Hom}_R(I, R)$.*

When I contains a non-zerodivisor on R, one can add the following.

(4) $I : I = R : I$.

4. Main results

For a Noetherian local ring $(A, \mathfrak{m})$, we set

$$\mathrm{Assh}\, A := \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \dim A/\mathfrak{p} = \dim A\} \subseteq \mathrm{Min}\, A \subseteq \mathrm{Ass}\, A.$$

Recall that A is *unmixed* (*quasi-unmixed*), if $\mathrm{Ass}\, \widehat{A} = \mathrm{Assh}\, \widehat{A}$ ($\mathrm{Min}\, \widehat{A} = \mathrm{Assh}\, \widehat{A}$).

Lemma 4.1

Let A be a Noetherian ring. Let $A \subseteq B \subseteq \mathrm{Q}(A)$ be a subring of $\mathrm{Q}(A)$ s.t. B is a finitely generated A-module. Let I be an ideal of B s.t. $I \subseteq A$ and $\mathrm{ht}_A I \geq 2$. If A is locally quasi-unmixed and B satisfies (S_2), then I is a trace ideal in A and $B = I : I$.

(Proof) Since A is locally quasi-unmixed, $\mathrm{ht}_B\, P = \mathrm{ht}_A(P \cap A)$ for $\forall P \in \mathrm{Spec}\, B$, so that $\mathrm{ht}_B\, I \geq 2$, whence $\mathrm{grade}_B\, I \geq 2$ because B satisfies (S_2). Therefore, $B = I : I$, so that

$$A : I \subseteq B : I = B = I : I \subseteq A : I.$$

This implies $I : I = A : I$. Hence I is a trace ideal in A.

Proposition 4.2

Let A be a Noetherian local ring and I $(\neq A)$ an ideal of A with $\mathrm{ht}_A I \geq 2$. Assume that $\exists\, \mathrm{K}_A$ and there exists an exact sequence

$$0 \to A \to \mathrm{K}_A \to C \to 0 \quad s.t. \quad IC = (0).$$

Then the following assertions hold true.

(1) $\widetilde{A} \cong \mathrm{K}_A$ *as an A-module.*

(2) $\mathrm{Hom}_A(\widetilde{A}, \mathrm{K}_A) \cong \widetilde{A}$ *as an $\widetilde{A}$-module.*

(3) *If K_A is a CM A-module, then $\widetilde{A}$ is a Gorenstein ring.*

(4) *If I is a trace ideal in A, then $\widetilde{A} = I : I$.*

(Proof) (1), (2) Let $\mathfrak{p} \in \mathrm{Ass}\, A$. Then $I \not\subseteq \mathfrak{p}$ because $\mathfrak{p} \in \mathrm{Ass}_A \mathrm{K}_A = \mathrm{Assh}\, A$, so that $C_{\mathfrak{p}} = (0)$ and

$$A_{\mathfrak{p}} \cong (\mathrm{K}_A)_{\mathfrak{p}} \cong \mathrm{K}_{(A_{\mathfrak{p}})}\,.$$

Hence $\mathrm{Q}(A)$ is a Gorenstein ring.

We choose

an A-submodule K of $\mathrm{Q}(A)$ s.t. $K \cong \mathrm{K}_A$ as an A-module.

Thus $\mathrm{Q}(A)K = \mathrm{Q}(A)$. By taking the K-dual of $0 \to A \xrightarrow{\varphi} K \to C \to 0$, we get

$$\psi : K : K = \mathrm{Hom}_A(K, K) \xrightarrow[\cong]{\varphi^*} \mathrm{Hom}_A(A, K) = K \quad \text{where} \quad \psi(1) = \varphi(1).$$

Hence, $\widetilde{A} \cong \mathrm{K}_A$ as an A-module, because $\widetilde{A} = K : K$ by Aoyama-Goto. Besides, since $\psi(1) = \varphi(1)$, $(K : K)/A \cong C$ as an A-module, so that

$$I\widetilde{A} = I(K : K) \subseteq A.$$

Notice that $\widetilde{A} \cong K$ as an $\widetilde{A}$-module, so $K = \alpha\widetilde{A}$ for some invertible $\alpha \in \mathrm{Q}(A)$. Therefore

$$K : \widetilde{A} = \alpha\widetilde{A} : \widetilde{A} = \alpha[\widetilde{A} : \widetilde{A}] = \alpha\widetilde{A} = K.$$

(3) As $\operatorname{depth}_A \widetilde{A} = \dim A$, every sop of A forms a regular sequence on $\widetilde{A}$, whence $\mathrm{ht}_{\widetilde{A}} M = \dim A$ for $\forall M \in \operatorname{Max} \widetilde{A}$, so that $\widetilde{A}_M$ is a Gorenstein ring.

(4) Suppose I is a trace ideal in A and set $B = I : I$. Since $I\widetilde{A} \subseteq A$, we have $\widetilde{A} \subseteq A : I = B$, while $B \subseteq \widetilde{A}$, since $IB = I \subseteq A$ and $\mathrm{ht}_A I \geq 2$. Thus $\widetilde{A} = I : I$. $\square$

Corollary 4.3

Let $(A, \mathfrak{m})$ *be a Noetherian local ring with* $d = \dim A$. *Let* $A \subseteq B \subseteq \mathrm{Q}(A)$ *be a subring of* $\mathrm{Q}(A)$ *s.t.* B *is a finitely generated* A*-module. We set* $\mathfrak{a} = A : B$ *and assume the following.*

(1) A *is a quasi-unmixed ring.*

(2) $\mathrm{ht}_A\, \mathfrak{a} \geq 2$.

(3) B *is a Gorenstein ring.*

Then the following assertions hold true.

(a) $B = \widetilde{A}$, $\mathrm{depth}_A\, B = d$, $\mathfrak{a}$ *is a trace ideal in* A, *and* $B = \mathfrak{a} : \mathfrak{a}$.

(b) $\exists\, \mathrm{K}_A$ *and* $\mathrm{K}_A \cong B$ *as an* A*-module.*

(c) $A = B$ *if and only if* A *is a CM local ring.*

(Proof) We have $B = \mathfrak{a} : \mathfrak{a}$ and $\mathfrak{a}$ is a trace ideal in A. Since $\mathrm{ht}_B\, M = \mathrm{ht}_A\, \mathfrak{m} = d$ for $\forall M \in \operatorname{Max} B$, every sop of A forms a regular sequence on B_M, so that it forms a regular sequence on B. Hence, $\operatorname{depth}_A B = d$.

Let $C = B/A$. Then $\dim_A C \le d-2$ since $\mathfrak{a}C = (0)$, so that

$$\mathrm{H}^d_{\mathfrak{m}}(A) \cong \mathrm{H}^d_{\mathfrak{m}}(B).$$

Therefore, $\mathrm{K}_{\widehat{A}} \cong \widehat{A} \otimes_A B$ as an $\widehat{A}$-module, whence

$$\exists\, \mathrm{K}_A \text{ and } \mathrm{K}_A \cong B \text{ as an } A\text{-module.}$$

We have $B \subseteq \widetilde{A}$ since $\mathrm{ht}_A\, \mathfrak{a} \ge 2$, while $\widetilde{A} \subseteq \widetilde{B} = B$. Hence, $B = \widetilde{A}$.

Suppose A is a CM ring. Then $\operatorname{depth}_A C \ge d-1$, which forces $C = (0)$ because $\dim_A C \le d-2$. Hence, $A = B$. □

Theorem 4.4 (Main theorem)

Let A be a Noetherian local ring with $d = \dim A \geq 2$. Suppose that A is quasi -unmixed. Let I $(\neq A)$ be an ideal of A with $\mathrm{ht}_A I \geq 2$ and $I \cap W(A) \neq \emptyset$. Set $B = I : I$. Then TFAE.

(1) *B is a Gorenstein ring.*

(2) *$\exists\, \mathrm{K}_A$ and I is a trace ideal in A s.t. (i) K_A is a CM A-module and (ii) there exists an exact sequence*

$$0 \to A \to \mathrm{K}_A \to C \to 0 \quad s.t. \quad IC = (0).$$

(3) *$\mathrm{depth}_A B = d$, $\exists\, \mathrm{K}_A$, and $B \cong \mathrm{K}_A$ as an A-module.*

When this is the case, A is unmixed, $B = \widetilde{A}$, and $\mathfrak{a} = A : B$ is a trace ideal in A with $B = \mathfrak{a} : \mathfrak{a}$.

Corollary 4.5

Let A be a Noetherian local ring with $d = \dim A \geq 2$. Let I $(\neq A)$ be an ideal of A with $\mathrm{ht}_A I \geq 2$ and $I \cap W(A) \neq \emptyset$. Set $B = I : I$. Then TFAE.

(1) *B is a Gorenstein ring, A is a homomorphic image of a CM ring, and $\mathrm{Min}\, A = \mathrm{Assh}\, A$.*

(2) *$\exists\, \mathrm{K}_A$ and I is a trace ideal in A s.t. (i) K_A is a CM A-module and (ii) there exists an exact sequence*

$$0 \to A \to \mathrm{K}_A \to C \to 0 \quad s.t. \quad IC = (0).$$

(3) *$\mathrm{depth}_A B = d$, $\exists\, \mathrm{K}_A$, and $B \cong \mathrm{K}_A$ as an A-module.*

When this is the case, A is unmixed, $B = \widetilde{A}$, and $\mathfrak{a} = A : B$ is a trace ideal in A with $B = \mathfrak{a} : \mathfrak{a}$.

5. Gorenstein Rees algebras

Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A \geq 2$ and $t = \operatorname{depth} A \geq 1$. For each ideal I of A, we set

$$\mathcal{R}_A(I) = A[It] = \sum_{n \geq 0} I^n t^n \subseteq A[t]$$

and call it the Rees algebra of I.

Example 5.1 (Hochster-Roberts)

Let $A = k[[x^2, y, x^3, xy]] \subseteq k[[x, y]]$ and $Q = (x^2, y)$. Then A is not CM, but

$$\mathcal{R}_A(Q^2) = A[Q^2 t] = A[x^4 t, x^2 yt, y^2 t] \subseteq A[t]$$

is a Gorenstein ring.

Theorem 5.2 (Shimoda)

Suppose $\dim A = 2$ *and* $\operatorname{depth} A = 1$*. Let* $Q = (a, b)$ *be a parameter ideal of* A*. Then TFAE.*

(1) $\mathcal{R}_A(Q^2)$ *is a Gorenstein ring.*

(2) (a) $a, b \in W(A)$,
(b) $[(a) : b] \cap [(b) : a] = (a) \cap (b)$, *and*
(c) $A/[(ab) + a[(a) : b] + b[(b) : a]]$ *is a Gorenstein ring.*

Question 5.3

Let Q be a parameter ideal of A. When is $\mathcal{R}_A(Q^d)$ a Gorenstein ring?

- If A is a CM local ring, then $\mathcal{R}_A(Q^d)$ is NOT a Gorenstein ring. (Goto-Nishida, Goto-Shimoda, Ikeda)

Theorem 5.4 (Goto-Iai)

Suppose $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ *for* $\forall i \notin \{1, d\}$ *and* $\mathrm{H}^1_{\mathfrak{m}}(A)$ *is a finitely generated* A*-module. Let* $Q = (a_1, a_2, \ldots, a_d)$ *be a parameter ideal of* A*. Then TFAE.*

(1) $\mathcal{R}_A(Q^d)$ *is a Gorenstein ring.*

(2) $\mathrm{H}^1_{\mathfrak{m}}(A) \neq (0)$, $\mathrm{r}_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = 1$, *and* $(0) :_A \mathrm{H}^1_{\mathfrak{m}}(A) = \sum_{i=1}^{d} \mathrm{U}(a_i A)$.

When this is the case, $\widetilde{A}$ *is a Gorenstein ring.*

Theorem 5.5

Let $(A, \mathfrak{m})$ be a Noetherian complete local ring s.t. $d = \dim A \geq 2$, $\operatorname{depth} A = 1$, and $\operatorname{Min} A = \operatorname{Assh} A$. Let I be an $\mathfrak{m}$-primary ideal of A. We set $B = I : I$ and $\mathfrak{a} = A : B$, and assume the following conditions.

(1) *B is a Gorenstein ring.*

(2) *$A \neq B$ and $\mathrm{r}_A(B/A) = 1$*

(3) *$\mathfrak{a} = (a_1, a_2, \ldots, a_d)B$ for some $a_1, a_2, \ldots, a_d \in \mathfrak{m}$.*

Then, $B = \mathfrak{a} : \mathfrak{a}$ and $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $Q = (a_1, a_2, \ldots, a_d)$.

(Proof) We have $B = \mathfrak{a} : \mathfrak{a}$. As $I{\cdot}(B/A) = (0)$, applying $\mathrm{H}^i_{\mathfrak{m}}(*)$ to the sequence

$$0 \to A \to B \to B/A \to 0$$

we get $\mathrm{H}^1_{\mathfrak{m}}(A) \cong B/A$ and $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for $\forall i \notin \{1, d\}$. Hence

$$(0) :_A \mathrm{H}^1_{\mathfrak{m}}(A) = \mathfrak{a} \quad \text{and} \quad \mathrm{r}_A(\mathrm{H}^1_{\mathfrak{m}}(A)) = 1.$$

Since $\operatorname{depth}_A B = d$ and $a_1, a_2, \ldots, a_d$ forms a system of parameters in A, the sequence $a_1, a_2, \ldots, a_d$ is B-regular. Hence

$$a_i \in W(A) \text{ and } a_i B \subseteq A \text{ for } 1 \leq \forall i \leq d$$

so that $B = \widetilde{A} \subseteq \frac{A}{a_i}$, whence $B = \frac{\mathrm{U}(a_i A)}{a_i}$. Hence

$$\mathfrak{a} = \sum_{i=1}^{d} a_i B = \sum_{i=1}^{d} \mathrm{U}(a_i A).$$

Consequently, $\mathcal{R}_A(Q^d)$ is a Gorenstein ring. □

6. Examples

Let

- $(S, \mathfrak{n})$ a Gorenstein complete local ring with $d = \dim S \geq 2$
- S contains a field k
- $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$ a parameter ideal of S s.t. $\mathfrak{q} \neq \mathfrak{n}$
- $A = k + \mathfrak{q}$

Then, A is a subring of S, and $\mathfrak{q}$ is a maximal ideal in A. We have

$$\ell_A(S/A) = \ell_A(S/\mathfrak{q}) - 1 < \infty.$$

Therefore, S is a finitely generated A-module, so that

A is a Noetherian complete local ring with $\dim A = d$ and $\operatorname{depth} A = 1$.

Theorem 6.1

If $\ell_S(S/\mathfrak{q}) = 2$, then $\mathcal{R}_A(Q^d)$ is a Gorenstein ring, where $Q = (a_1, a_2, \ldots, a_d)A$.

Example 6.2

Let $S = k[[X_1, X_2, \ldots, X_d]]$ $(d \geq 2)$ and $\mathfrak{q} = (X_1^2, X_2, \ldots, X_d)S$. Then

$$\mathcal{R}_A(Q^d) \text{ is a Gorenstein ring}$$

where $A = k + \mathfrak{q}$ and $Q = (X_1^2, X_2, \ldots, X_d)A$.

Let

- $B = k[[t, s]]$ the formal power series ring over a field k
- $P = k[[H]] \subsetneq V = k[[t]]$, where H is a symmetric numerical semigroup
- $\mathfrak{c} = P : V = t^c V$, $0 < c \in H$
- $A = P + sB \subseteq B$
- $\mathfrak{a} = A : B = \mathfrak{c} + sB = (t^c, s)B$

Then

A is a Noetherian complete local ring with $\dim A = 2$ and $\operatorname{depth} A = 1$.

We set $Q = (t^c, s)$. Then, Q is a parameter ideal of A.

Theorem 6.3

The Rees algebra $\mathcal{R}_A(Q^2)$ is a Gorenstein ring.

Let

- S a Gorenstein complete local ring with $d = \dim S \geq 2$
- $\mathfrak{q} = (a_1, a_2, \ldots, a_d)$ a parameter ideal of S
- $A = S \times_{S/\mathfrak{q}} S = \{(x, y) \in S \times S \mid x \equiv y \mod \mathfrak{q}\}$

Then

$$A \text{ is a Noetherian complete local ring with } \dim A = d \text{ and depth } A = 1$$

Let $\alpha_i = (a_i, a_i) \in A$ for each $1 \leq i \leq d$ and set $Q = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. Then, Q is a parameter ideal of A.

Theorem 6.4

The Rees algebra $\mathcal{R}_A(Q^d)$ is a Gorenstein ring.

Example 6.5

Let $U = k[[X_1, X_2, \ldots, X_d, Y_1, Y_2, \ldots, Y_d]]$ $(d \geq 2)$ and set

$$A = U/[(X_1, X_2, \ldots, X_d) \cap (Y_1, Y_2, \ldots, Y_d)] \cong S \times_k S$$

where $S = k[[X_1, X_2, \ldots, X_d]]$.

For each $1 \leq i \leq d$, let z_i denote the image of $X_i + Y_i$ in A. Then

$$\mathcal{R}_A(Q^d) \text{ is a Gorenstein ring}$$

where $Q = (z_1, z_2, \ldots, z_d)$.

Thank you for your attention.